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AN ALGEBRAIC DECOMPOSITION OF HEMITROPIC PSEUDOTENSORS IN N DIMENSIONS AND APPLICATIONS TO MICROPOLAR CONTINUUM THEORIES*

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The paper is devoted to problems related to algebraic decompositions and coordinate representations of tensors with constant components, hemitropic tensors and pseudotensors. Such objects of tensor algebra are interesting from the micropolar continuum mechanics viewpoint. Algebraic properties and coordinate representations of tensors and pseudotensors with constant components are discussed. The base examples of tensors with constant components usable in continuum mechanics are given. An algebraic algorithm for coordinate representations of tensors and pseudotensors with constant components proposed by prof. G.B. Gurevich is highlighted and employed.The notions of fully isotropic, conventionally isotropic, unconventionally isotropic, semi-isotropic (demitropic, hemitropic, chiral) fourth rank tensors and pseudotensors are proposed. The coordinate representations of a Cartesian hemitropic fourth rank tensor in three-dimensional space are obtained in terms of the Kronecker delta and metric tensor products. Based on an unconventional definition of a hemitropic fourth rank tensor, general coordinate representations in dimensions in terms of the Kronecker deltas and metric tensors are given. A comparison of an arbitrary hemitropic fourth rank tensor and a tensor with constant components is carried out. A general form of the elastic potential of a linear anisotropic micropolar elastic continuum is obtained by the pseudotensor technique. Obtained anisotropic micropolar potential is reduced to a hemitropic one by proposed coordinate representations of fourth rank tensors. Coordinate representations for constitutive tensors and pseudotensors usable in mathematical modeling of linear hemitropic micropolarelastic continua are obtained as a modification of pseudo-tensors with constant components in three-dimensional space.

Keywords: pseudotensor, tensor with constant components, constitutive pseudotensor, fundamental orienting pseudoscalar, chiral, micropolar hemitropic continuum.

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1. Introduction and prerequisites

The constitutive fourth rank tensors and pseudotensors are required for a reasonable development of mathematical models of linear anisotropic micropolar elastic media [1–13]. Thus, the algebraically consistent derivation of the constitutive equations of micropolar continua should be provided by the formalism of pseudotensor algebra [14–20]. In this case, a significant role is given to the rotational invariance (demitropy, hemitropy) of tensors and pseudotensors [21–25]. In-depth study of the pseudotensor formalism can be found in monographs and papers on tensor analysis and continuum mechanics [14–20, 25–34]. These are to be easily obtained via literary search. The paper is aimed at investigation of properties of fourth rank tensors and their coordinate representations and decompositions interesting from the viewpoint of mechanics of micropolar continua [35–43].

The paper is organised as follows. After the Introduction, in Sec. 2, properties and coordinate representations of tensors and pseudotensors with constant components in N-dimensional Euclidean spaces are recalled and discussed. Spaces of mathematical dimensions higher than 3 can be applied for modeling materials with complex microstructure (for instance, growing solids [37]). The base examples of tensors with constant components usable in continuum mechanics are given. An algebraic algorithm for coordinate representations of tensors and pseudotensors with constant components proposed by prof. G.B. Gurevich in [17] is highlighted and employed.

In Sec. 3, the unconventional terminology consistent to tensor algebra frameworks is introduced. This part of study is to be considered as a refinement of tensor algebra theory. The notions of fully isotropic, conventionally isotropic, unconventionally isotropic, semi-isotropic (demitropic, hemitropic, chiral) fourth rank tensors and pseudotensors are proposed. The coordinate representations of a Cartesian hemitropic fourth rank tensor in three dimensional space are obtained in terms of the Kronecker delta and metric tensor products. A hemitropic fourth rank tensor is splitted into two additive parts: a tensor with constant component and tensor product of the Kronecker deltas.

The Sec. 4 is devoted to coordinate representations for constitutive fourth rank tensors and pseudotensors arising in mathematical modeling of linear hemitropic micropolar continua. Those are given in terms of the metric tensor and the constitutive hemitropic pseudoinvariants in three dimensional spaces. The constitutive hemitropic invariants can be easily replaced by constitutive mechanical constants listed by G, ν, L, …

An unconventional terminology is used throughout the paper. Its meaning will be elucidated in Sec. 3.

2. Tensors and pseudotensors with constant components

A tensor (pseudotensor) with constant components considered by G.B. Gurevich in [17, p. 170] is a tensor (pseudotensor) that retains all its components unaltered under any linear transformations of the coordinate frame: the most important of them are rotations, scaling, central inversion, mirror reflections.

An absolute second rank tensor with constant components coincides with the unit affinor up to a constant factor a (absolute invariant):

$$C_k^h = a\delta_k^h. \tag{1}$$

There are two types of third rank pseudotensors with constant components. They are proportional to the permutation symbols (of weights +1 and –1):

$$\overset{[+1]}{C}{}^{hks} = a\overset{[+1]}{\epsilon}{}^{hks}, \quad \overset{[-1]}{C}_{hks} = b\overset{[-1]}{\epsilon}_{hks}, \tag{2}$$

where a and b are the absolute invariants.

It should be also noted that there are no absolute third rank tensors with constant components. The Kronecker deltas and permutation symbols are the base and the most important examples of tensors with constant components.

It is easy to show that a general absolute fourth rank tensor C^{il}_{sm} with constant components can be represented as linear combinations of δ-symbols tensor products according to

$$C^{il}_{sm} = a\delta^i_s\delta^l_m + c\delta^l_s\delta^i_m, \tag{3}$$

where a and c are absolute invariants (absolute scalars)[1].

Equation (3) is valid in any frame in Cartesian coordinates and can be represented as follows

$$C_{ilsm} = a\delta_{is}\delta_{lm} + c\delta_{ls}\delta_{im}. \tag{4}$$

G.B. Gurevich proposed a general algorithm for obtaining tensors and pseudotensors with constant components for positive (or negative) integer weights (see [17, p. 170–182]). For example, the general form of the pseudotensor $\overset{[w]}{C}{}^{h_1h_2\ldots h_s}_{k_1k_2\ldots k_r}$ with constant components of a negative integer weight is obtained as

$$\overset{[w]}{C}{}^{h_1h_2\ldots h_s}_{k_1k_2\ldots k_r} = \sum_{P=1}^{r!} \lambda_P \delta^{h_1}_{\{k_1}\delta^{h_2}_{k_2}\ldots\delta^{h_s}_{k_s} \underbrace{\overset{[-1]}{\epsilon}_{k_{s+1}\ldots k_{s+N}} \ldots \overset{[-1]}{\epsilon}_{k_{r-N+1}\ldots k_r\}_P}}_{|w|}, \tag{5}$$

where r is number of covariant indices, s is number of contravariant indices, N denotes space dimension, w is weight (negative integer number), λ_P are constants (absolute invariants), $P = 1, 2, \ldots, r!$ is a permutation of a series

$$k_1, \ldots, k_s, \ldots, k_{s+N}, \ldots, k_{r-N+1}, \ldots, k_r.$$

In the equation (5) all permutations are performed on covariant indices enclosed in curly braces.

The number of covariant, contravariant indices and the weight of the pseudotensor must satisfy the equation

$$r = s + N|w|, \tag{6}$$

and, consequently, is valid while

$$r \geq s. \tag{7}$$

If the equation (6) is violated, then the pseudotensor $\overset{[w]}{C}{}^{h_1h_2\ldots h_s}_{k_1k_2\ldots k_r}$ with constant components turns to the zero one.

3. Absolute hemitropic fourth rank tensors

Starting from Sec. 3 an unconventional terminology is used. Consider Euclidean space of three dimensions. Unconventionally isotropic (fully isotropic) tensor (pseudotensor) is a one that retains its components for any rotation of the coordinate frame, mirror reflections and central inversions of three-dimensional space [21, 22, 24, 25].

[1] It is clear that components C^{12}_{12} and C^{12}_{21} themselves are absolute invariants.

A conventionally isotropic tensor (pseudotensor) is a one that insensitive to the coordinate frame rotations. Conventionally isotropic tensors (pseudotensors) are meaningfully called in the present study as demitropic, hemitropic or semi-isotropic.

For an absolute hemitropic fourth rank tensor in Cartesian coordinates, the following representation in terms of δ-symbols holds [21, p.77]

$$H_{islm} = a\delta_{is}\delta_{lm} + \frac{b-c}{2}(\delta_{il}\delta_{sm} - \delta_{im}\delta_{sl}) + \frac{b+c}{2}(\delta_{il}\delta_{sm} + \delta_{im}\delta_{sl}), \tag{8}$$

or

$$H_{islm} = a\delta_{is}\delta_{lm} + b\delta_{il}\delta_{sm} + c\delta_{im}\delta_{sl}. \tag{9}$$

Here a, b, c are the rotational invariants insensitive to the coordinate frame rotations. In contrast to the coordinate representation (4) for a tensor with constant components, the representation (9) for hemitropic fourth rank tensors involves the additive term $\delta_{il}\delta_{sm}$ multiplied by b. In virtue of (4), an absolute hemitropic fourth rank tensor can be decomposed on tensor with constant components as follows

$$H_{islm} = C_{islm} + b\delta_{il}\delta_{sm}. \tag{10}$$

The coordinate representation (9) can be easily obtained in an arbitrary coordinate net by replacing the δ-symbols with metric tensors:

$$H^{islm} = ag^{is}g^{lm} + bg^{il}g^{sm} + cg^{im}g^{sl}. \tag{11}$$

Since the components of the metric tensor g^{is} are insensitive to the coordinate frame rotations, then the right-hand side in (11) also remains unchanged. If a, b, c are to be absolute invariants, then the right-hand side in (11) is also unaltered under mirror reflections and central inversions.

Obviously, a fully isotropic fourth rank tensor cannot be discriminated from a hemitropic tensor.

4. Application to linear micropolar elasticity

Fourth rank tensors and pseudotensors play a significant role in mathematical models of linear anisotropic micropolar elastic continua [1–13, 35, 39].

Taking apart thermomechanics of continua let us focus our attention to the micropolar elastic potential $\mathcal{U}$ (per unit invariant volume) with natural pseudotensor arguments

$$\mathcal{U} = \mathcal{U}(\epsilon_{ij}, \overset{[+1]}{\kappa_{i\cdot}^{\cdot s}}), \tag{12}$$

where ϵ_{ij} is the asymmetric strain tensor; $\overset{[+1]}{\kappa_{i\cdot}^{\cdot s}}$ is the wryness pseudotensor of positive weight +1. The elastic potential $\mathcal{U}$ is introduced as an absolute invariant (scalar), which does not depend on mirror reflections and the central inversion of a three-dimensional space.

In the case of a linear anisotropic micropolar elastic solid, the elastic potential in an arbitrary coordinate system is obtained in the form:

$$\mathcal{U} = \underset{1}{H}^{islm}\epsilon_{is}\epsilon_{lm} + \underset{2}{\overset{[-2]}{H_{\cdot s \cdot m}^{i \cdot l \cdot}}}\overset{[+1]}{\kappa_{i\cdot}^{\cdot s}}\overset{[+1]}{\kappa_{l\cdot}^{\cdot m}} + \underset{3}{\overset{[-1]}{H_{\cdot\cdot\cdot m}^{isl\cdot}}}e_{is}\overset{[+1]}{\kappa_{l\cdot}^{\cdot m}}. \tag{13}$$

Revisiting representations (11) from viewpoint of Sec. 3 let us transform the elastic potential (13) to following form:

$$\mathcal{U} = \underset{1}{H}^{islm}\epsilon_{is}\epsilon_{lm} + \overset{[-2]}{\underset{2}{H}}{}^{islm}\overset{[+1]}{\kappa}_{is}\overset{[+1]}{\kappa}_{lm} + \overset{[-1]}{\underset{3}{H}}{}^{islm}\epsilon_{is}\overset{[+1]}{\kappa}_{lm}\ . \tag{14}$$

Note that the only constitutive fourth rank tensor that is sensitive to mirror reflections and the central inversions of three-dimensional space is the constitutive pseudotensor $\overset{[-1]}{\underset{3}{H}}{}^{islm}$. A micropolar solid is called as hemitropic if the components of all the constitutive tensors retain under the coordinate frame rotations, i.e. they are hemitropic.

Taking account of the results of the previous section, we need to transform the energy form (14) with the fundamental orienting pseudoscalar e thus eliminating the pseudotensors weights[2]:

$$\mathcal{U} = \underset{1}{H}^{islm}\epsilon_{is}\epsilon_{lm} + e^2\overset{[-2]}{\underset{2}{H}}{}^{islm}\frac{\overset{[+1]}{\kappa}_{is}}{e}\frac{\overset{[+1]}{\kappa}_{lm}}{e} + e\overset{[-1]}{\underset{3}{H}}{}^{islm}\epsilon_{is}\frac{\overset{[+1]}{\kappa}_{lm}}{e}, \tag{15}$$

finally come to the conventional absolute tensor representation

$$\mathcal{U} = \underset{1}{H}^{islm}\epsilon_{is}\epsilon_{lm} + \underset{2}{H}^{islm}\kappa_{is}\kappa_{lm} + \underset{3}{H}^{islm}\epsilon_{is}\kappa_{lm}. \tag{16}$$

The obtained form of elastic potential (16) is usually used in developments of models of hemitropic micropolar elastic continua. By using the coordinate representations in terms of metric tensor (11) for constitutive pseudotensors of a linear hemitropic micropolar elastic continuum, one can obtain in the following forms

$$\begin{aligned}
\underset{1}{H}^{islm} &= \underset{1}{a}\,g^{is}g^{lm} + \underset{1}{b}\,g^{il}g^{sm} + \underset{1}{c}\,g^{im}g^{sl},\\
\underset{2}{H}^{islm} &= \underset{2}{a}\,g^{is}g^{lm} + \underset{2}{b}\,g^{il}g^{sm} + \underset{2}{c}\,g^{im}g^{sl},\\
\underset{3}{H}^{islm} &= \underset{3}{a}\,g^{is}g^{lm} + \underset{3}{b}\,g^{il}g^{sm} + \underset{3}{c}\,g^{im}g^{sl}.
\end{aligned} \tag{17}$$

Here $\underset{\mathfrak{a}}{a}, \underset{\mathfrak{a}}{b}, \underset{\mathfrak{a}}{c}$ $(\mathfrak{a} = 1, 2, 3)$ are nine constitutive constants of a hemitropic micropolar elastic solid. From the viewpoint of tensor algebra these constitutive constants are at least hemitropic invariants.

Returning to the energy form (14) can be attained by the transformation of representations for the constitutive tensors (17) making the following replacements

$$\begin{aligned}
&\overset{[-2]}{\underset{2}{H}}{}^{islm} = e^{-2}\underset{2}{H}^{islm}, \quad \overset{[-1]}{\underset{3}{H}}{}^{islm} = e^{-1}\underset{3}{H}^{islm},\\
&\overset{[-2]}{\underset{2}{a}} = e^{-2}\underset{2}{a}, \quad \overset{[-2]}{\underset{2}{b}} = e^{-2}\underset{2}{b}, \quad \overset{[-2]}{\underset{2}{c}} = e^{-2}\underset{2}{c},\\
&\overset{[-1]}{\underset{3}{a}} = e^{-1}\underset{3}{a}, \quad \overset{[-1]}{\underset{3}{b}} = e^{-1}\underset{3}{b}, \quad \overset{[-1]}{\underset{3}{c}} = e^{-1}\underset{3}{c}.
\end{aligned} \tag{18}$$

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[2] The fundamental orienting pseudoscalar e of weight +1 in N-dimensional Euclidean spaces can be defined as the skew product [44, p. 63–65] of covariant base vectors $\underset{1}{\boldsymbol{\imath}}, \underset{2}{\boldsymbol{\imath}}, \ldots, \underset{N}{\boldsymbol{\imath}}$:

$$\lceil \underset{1}{\boldsymbol{\imath}}, \underset{2}{\boldsymbol{\imath}}, \ldots, \underset{N}{\boldsymbol{\imath}} \rfloor = e.$$

It can be shown that the metric tensor determinant g is a pseudoscalar of weight +2 and $e = \sqrt{g}$ for a right-handed frame and $e = -\sqrt{g}$ for a left-handed frame.

In three-dimensional Euclidian spaces fundamental orienting pseudoscalar e is defined by the triple product $e = \underset{1}{\boldsymbol{\imath}} \cdot (\underset{2}{\boldsymbol{\imath}} \times \underset{3}{\boldsymbol{\imath}})$.

Upon substituting the equations (18) into the coordinate representations (17), we get

$$
\begin{aligned}
\underset{1}{H}^{islm} &= \underset{1}{a}\, g^{is} g^{lm} + \underset{1}{b}\, g^{il} g^{sm} + \underset{1}{c}\, g^{im} g^{sl}, \\
\overset{[-2]}{\underset{2}{H}}{}^{islm} &= \overset{[-2]}{\underset{2}{a}}\, g^{is} g^{lm} + \overset{[-2]}{\underset{2}{b}}\, g^{il} g^{sm} + \overset{[-2]}{\underset{2}{c}}\, g^{im} g^{sl}, \qquad (19) \\
\overset{[-1]}{\underset{3}{H}}{}^{islm} &= \overset{[-1]}{\underset{3}{a}}\, g^{is} g^{lm} + \overset{[-1]}{\underset{3}{b}}\, g^{il} g^{sm} + \overset{[-1]}{\underset{3}{c}}\, g^{im} g^{sl}.
\end{aligned}
$$

The constitutive hemitropic invariants $\underset{\alpha}{a}$, $\overset{[-1]}{\underset{\alpha}{b}}$, $\overset{[-2]}{\underset{\alpha}{c}}$ ($\alpha = 1, 2, 3$) in representations (19) can be replaced by conventional mechanical constants such as G, ν, L, … In this case the characteristic microlength L is revealed as a pseudoscalar of negative weight -1, whereas G, ν are the absolute invariants.

Conclusions

The paper concerns problems related to the coordinate representations of tensors and pseudotensors with constant components, absolute hemitropic tensors and their application to the mechanics of hemitropic micropolar solids.

1. Properties and coordinate representations of tensors and pseudotensors with constant components have been discussed.

2. A more consistent unconventional terminology related to the notions of fully isotropic (unconventionally isotropic), conventionally isotropic, chiral, demitropic, hemitropic absolute tensors has been proposed.

3. A coordinate representation for a hemitropic fourth rank tensor has been obtained in terms of metric tensor.

4. The forms of additive representations of hemitropic absolute fourth rank tensors in terms of absolute fourth rank tensors with constant components have been obtained and highlighted.

5. A general form of the elastic potential of a linear anisotropic micropolar elastic continuum has been obtained by the pseudotensor technique. Obtained anisotropic micropolar potential has been reduced to a hemitropic one by proposed coordinate representations of fourth rank tensors.

6. Coordinate representations for the constitutive fourth rank hemitropic tensors and pseudotensors have been given. Those are true for the material scalars and pseudoscalars. It has been noted that the characteristic microlength L reveals as a pseudoscalar of weight -1.

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АЛГЕБРАИЧЕСКОЕ РАЗЛОЖЕНИЕ ГЕМИТРОПНЫХ ПСЕВДОТЕНЗОРОВ В N-МЕРНЫХ ПРОСТРАНСТВАХ И ПРИЛОЖЕНИЕ К ТЕОРИЯМ МИКРОПОЛЯРНЫХ КОНТИНУУМОВ*

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Рассматривается декомпозиция и координатные представления тензоров с постоянными компонентами, полуизотропных тензоров и псевдотензоров. Такие объекты тензорной алгебры представляют интерес с точки зрения механики микрополярных континуумов. Обсуждаются свойства и способы координатного представления тензоров и псевдотензоров с постоянными компонентами. Приведены базовые примеры тензоров с постоянными компонентами, широко использующиеся в механике сплошных сред. Обсуждается алгебраический алгоритм координатных представлений тензоров и псевдотензоров с постоянными компонентами, предложенный проф. Г.Б. Гуревичем. Введены понятия полностью изотропных, условно изотропных, неконвенционально изотропных, полуизотропных (демитропных, гемитропных, хиральных) тензоров и псевдотензоров четвертого ранга. Получены декартовы координатные представления гемитропного тензора четвертого ранга в трехмерном пространстве в терминах символов Кронекера, метрических тензоров и их тензорных произведений. На основе неконвенционального определения полуизотропного тензора четвертого ранга приводится координатное представление в терминах символов Кронекера и метрических тензоров. Выясняются условия приведения произвольного полуизотропного тензора четвертого ранга к тензору с постоянными компонентами. Псевдотензорным методом получен общий вид упругого потенциала для линейного анизотропного микрополярного упругого континуума. Полученный анизотропный микрополярный упругий потенциал редуцируется к гемитропному с помощью предложенных координатных представлений тензоров четвертого ранга. Координатные представления для определяющих тензоров и псевдотензоров, использующихся при математическом моделировании линейных гемитропных микрополярных упругих континуумов, даны в терминах метрического тензора в трехмерном пространстве.

Ключевые слова: псевдотензор, тензор с постоянными компонентами, определяющий псевдотензор, фундаментальный ориентирующий псевдоскаляр, хиральный, микрополярный гемитропный континуум.

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